

Lecture 15.

Geometric and Negative Binomial Random Vars.

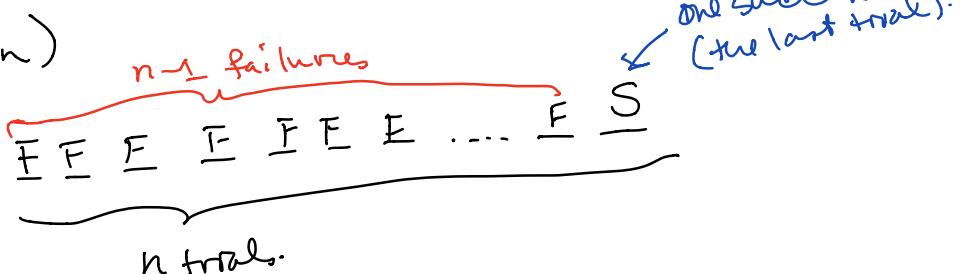
- Suppose we have a Bernoulli trial with
 - probability p of success.
- We define a discrete random variable X by
$$X = \# \text{ of trials until a successful trial}$$

That is, $X=1$ if the first trial is successful.
 $X=2$ if the first trial fails & the second is successful.

⋮

- range(X) = $1, 2, 3, 4, \dots$
- what is the probability mass function of X ?

- $P(X=n)$



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- Since the trials are independent, the probability of

this sequence appears \rightsquigarrow

$$P(X=n) = \underbrace{(1-p)^{n-1}}_{n-1 \text{ failure}} \underbrace{p}_{\text{one success at the end.}}$$

— Such a random variable is called "geometric".

- Note that

$$\begin{aligned} \sum_{n=1}^{\infty} P(X=n) &= \sum_{n=1}^{\infty} (1-p)^{n-1} p \\ &= p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \underbrace{\sum_{j=0}^{\infty} (1-p)^j}_{\substack{\text{converges} \\ \text{geometric series}}} = p \frac{1}{1-(1-p)} \\ &= \frac{p}{p} = 1. \end{aligned}$$

- so success eventually occurs with probability 1.
- for example, you start flipping a coin. You don't know when, but you know that you eventually get a heads.

- Note that if X is geometric with prob. p of

success

$$\begin{aligned} P(X \geq k) &= \sum_{n=k}^{\infty} (1-p)^{n-1} p \\ &= p \sum_{n=k}^{\infty} (1-p)^{n-1} = p \sum_{m=k-1}^{\infty} (1-p)^m \\ &= p (1-p)^{k-1} \sum_{j=0}^{\infty} (1-p)^j = p (1-p)^{k-1} \frac{1}{1-(1-p)} \\ &= (1-p)^{k-1} \end{aligned}$$

Ex: You roll a die. Success is a 1 or a 2, failure is anything else. What is the prob. that you roll 10 times and only see a 1 or 2 on the 10th roll?

$$p = 2/6 = 1/3.$$

$$P(X=10) = \left(\frac{2}{3}\right)^9 \left(\frac{1}{3}\right) = \frac{2^9}{3^{10}} \approx 0.008 \approx 0.8\%.$$

Negative Binomial Random Variable

We now consider a generalization of the geometric random variable.

Let $r \geq 1$ be some integer.

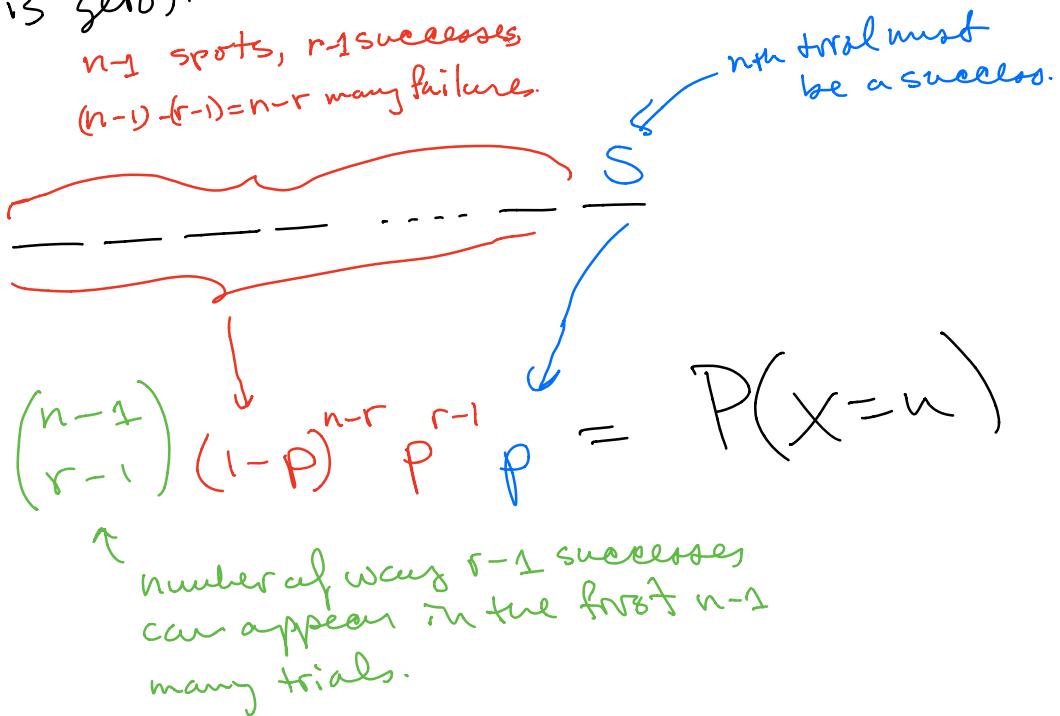
Let $r \geq 1$ be some integer. We assume we have some Bernoulli trial, with probability p of success. We say that X is negative

Binomial if

$X = \# \text{ of trials until } r \text{ successes.}$

In particular, X is geometric iff X is Negative Binomial with $r=1$.

Let's compute the probability mass function of X : Suppose we want $P(X=n)$ where $n \geq r$ (note that the probability of r successes in $n < r$ trials is zero).



So $P(X=n) = \binom{n-1}{r-1} (1-p)^{n-r} p^r$ for $n=r, r+1, r+2, \dots$

Let's compute the mean and variance of the negative binomial RV. Let $k \geq 1$
If X is negative binomial with parameter k , then

$$\begin{aligned}
 E[X^k] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
 &= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \quad \left[\text{use: } n \binom{n-1}{r-1} = r \binom{n}{r} \right] \\
 &= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-(r+1)} \\
 &= \frac{r}{p} E[(Y-1)^{k-1}]
 \end{aligned}$$

where Y is negative binomial with param $r+1$.

Letting $k=1$, we get

$$E[X] = \frac{r}{p} E[(Y-1)^0] = \frac{r}{p}.$$

For $\text{Var}(X)$, we have

$$\begin{aligned}
 \text{Var}(X) &= E[X^2] - E[X]^2 \\
 &= \frac{r}{p} E[(Y-1)^2] - \left(\frac{r}{p}\right)^2 \\
 &= \frac{r}{p} \left(\frac{r+1}{p} - 1\right) - \left(\frac{r}{p}\right)^2 \\
 &= \frac{r}{p} \left(\frac{r+1}{p} - \frac{p}{p} - \frac{r}{p}\right) \\
 &= \frac{r}{p} \left(\frac{1-p}{p}\right) = \frac{r(1-p)}{p^2}.
 \end{aligned}$$

In particular, for $r=1$, X is geometric and so we have $E[X] = \frac{1}{p}$, $\text{Var}(X) = \frac{(1-p)}{p^2}$.

So for example, if I flip a coin, how many times do I need to flip it before I get a heads? If $X = \#$ of flips until a head shows up, then $E[X] = \frac{1}{p} = \frac{1}{1/2} = 2$.

Some "hand-wavy" remarks:

- we may view a negative binomial RV X with parameters r, p as a sum of geometric random variables X_i with parameter p where $X_i = \#$ of flips until the i^{th} success.

Then
$$\begin{aligned} E[X] &= E[X_1 + \dots + X_r] \} \text{ next section} \\ &= \sum_{i=1}^r E[X_i] \} (4.4). \\ &= \sum_{i=1}^r \frac{1}{p} = \frac{r}{p}. \end{aligned}$$